Self-Similar Solutions, Critical Behavior and Convergence to Attractor in Gravitational Collapse

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Abstract

General relativity as well as Newtonian gravity admits self-similar solutions due to its scale-invariance. This is a review on these self-similar solutions and their relevance to gravitational collapse. In particular, our attention is mainly paid on the crucial role of self-similar solutions in the critical behavior and attraction in gravitational collapse.

1 Introduction

The basic equation of general relativity is the Einstein equations. Since the Einstein equations are simultaneous second-order partially differential equations (PDEs), it is not easy to obtain solutions. It is useful to assume symmetry such as spherical symmetry and staticity in obtaining solutions. However, it is often difficult to obtain inhomogeneous and dynamical solutions. In this context, it is quite useful to assume *self-similarity*. For example, in spherical symmetry, this assumption reduces the PDEs to a set of ordinarily differential equations (ODEs). Therefore, the assumption of self-similarity is very powerful in finding exact and/or numerical dynamical and inhomogeneous solutions. The application of self-similar solutions is very large including cosmological perturbations, star formation, gravitational collapse, primordial black holes, cosmological voids and cosmic censorship. Among them, the most widely researched model is the spherically symmetric self-similar system with perfect fluids, which was pioneered by by Cahill and Taub (1971) in general relativity and by Penston (1969) and Larson (1969) in Newtonian gravity.

In recent progress in general relativity, self-similar solutions have called attention not only because they are easy to obtain but also because they may play important roles in cosmological situations and/or gravitational collapse. One of the important roles of self-similar solutions is to describe asymptotic behaviors of more general non-self-similar solutions. This role has been shown explicitly in several models such as homogeneous cosmological models. In cosmological context, the suggestion that spherically symmetric fluctuations might naturally evolve from complex initial conditions via the Einstein equations to a self-similar form has been termed the 'similarity hypothesis' (Carr 1993, Carr and Coley 1999). The hypothesis has been also suggested to hold in more general situations, including gravitational collapse.

In the present article, we pay attention to a spherically symmetric system with a perfect fluid. For this system, recent numerical simulations of PDEs have revealed two interesting behaviors in gravitational collapse, one is the critical behavior and the other is the convergence to attractor. The critical behavior was observed around the threshold of black hole formation. There is a self-similar solution that sits at the threshold, which is called a critical solution. For near-critical collapse, the solution first approach the critical solution but diverges away from this solution after that. There is observed the scaling law for the formed black hole for supercritical collapse. This is observed only as a result of fine-tuning the parameter which parametrizes initial data. On the other hand, the convergence to attractor is observed without fine-tuning. The density and velocity fields around the center approach those of another self-similar solution in an approach to the formation of singularity at the center, which is called an attractor solution.

The main claim of this review article is that the critical behavior and convergence to attractor in gravitational collapse are both well understood in the context of self-similarity hypothesis and stability analysis in gravitational collapse. The difference between these two behaviors is in the stability of the

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pertinent self-similar solution. We also briefly review the recent work on how these two behaviors are in Newtonian gravity.

2 Self-similarity

In general relativity, self-similarity or *homothety* is covariantly defined in terms of the homothetic vector ξ which satisfies the homothetic Killing equation

$$\mathcal{L}_\xi g_{\mu\nu} = 2g_{\mu\nu}. \tag{1}$$

We consider a perfect fluid as matter field, the stress-energy tensor of which is given by

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}. \tag{2}$$

In a spherically symmetric spacetime with self-similarity, where the homothetic vector is neither parallel nor orthogonal to the four velocity of the fluid element, the line element is written in terms of the comoving coordinate r as (Cahill and Taub 1971)

$$ds^2 = -e^{\sigma(\xi)}dt^2 + e^{\omega(\xi)}dr^2 + r^2 S^2(\xi)(d\theta^2 + \sin^2\theta d\phi^2), \tag{3}$$

where $\xi = t/r$. In this coordinate system, the dimensionless quantity h such as e^σ, e^ω and S has scale-invariance in the following sense:

$$h(t, r) = h(at, ar) = h(\xi), \tag{4}$$

where $a > 0$ is an arbitrary constant. For a self-similar spacetime, the density field ρ is given via the Einstein equations by

$$\rho(t, r) = a^2 \rho(at, ar). \tag{5}$$

It implies that the density field is written as

$$\rho(t, r) = \frac{1}{8\pi r^2}\eta(\xi). \tag{6}$$

Therefore, the time evolution is equivalent with the scale transformation. An example of the time evolution of the density field is displayed in Fig. 1.
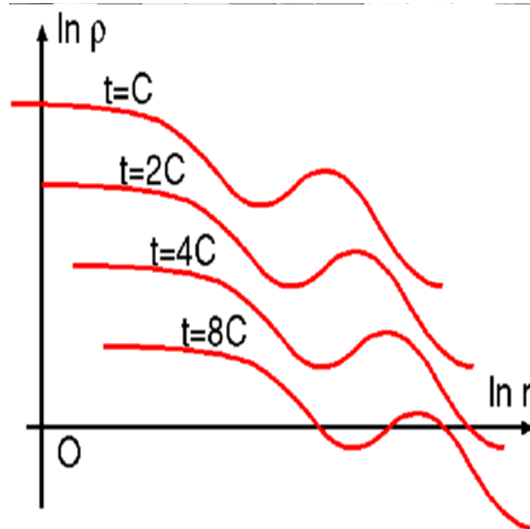

Figure 1: Schematic picture for the self-similar evolution of the density field.

The above definition of self-similarity can be generalized. One possibility is to discretize the above definition: the dimensionless quantity h satisfies

$$h(t, r) = h(at, ar), \quad a = e^{n\Delta}, \tag{7}$$

where $n = 0, \pm 1, \pm 2, \cdots$. This kind of self-similarity is referred to as *discrete self-similarity*. The discrete self-similar solution has been found to be important in the critical behavior observed in the gravitational collapse of a massless scalar field (Choptuik 1993) and gravitational waves (Abrahams and Evans 1993). In this context, the originally defined self-similarity is referred to as *continuous self-similarity*.

Another possible generalization is to allow the dimensionless quantity h to be a function of r/t^α. The generalization along this direction was done by Carter and Henriksen (1989,1991) and is referred to as *kinematic self-similarity*. The kinematic self-similarity is covariantly defined by the existence of kinematic self-similarity vector, which satisfies

$$\mathcal{L}_\xi g_{\mu\nu} = 2\delta g_{\mu\nu}, \quad \mathcal{L}_\xi u_\mu = \alpha u_\mu, \tag{8}$$

where δ and α are arbitrary constants. See Coley (1997), Benoit and Coley (1998) and Maeda et al. (2002a, 2002b, 2003) for more details of kinematic self-similarity and kinematic self-similar solutions. In this context, the originally defined self-similarity is referred to as *first-kind self-similarity*.

3   Self-similar solutions

Hereafter we mainly concentrate on the originally defined self-similarity and refer to it simply as self-similarity unless it is confusing. For a perfect fluid, the Einstein equation requires that the equation of state be given by the following form:

$$P = k\rho, \tag{9}$$

where k is constant (Cahill and Taub 1971). The fluid is referred to as a dust fluid for $k = 0$, a radiation fluid for $k = 1/3$ and a stiff matter for $k = 1$. There are infinitely many self-similar solutions. These solutions are obtained by solving ODEs. The classification of self-similar solutions of this system has been done in the homothetic approach by Goliath, Nilsen and Uggla (1998a, 1998b) and in the comoving approach by Carr and Coley (2000a, 2000b). For the complementarity of these two approaches, see Carr et al. (2001). For a brief review of self-similar solutions and, in particular, the classifications of them, see Carr and Coley (1999).

In obtaining physical solutions, it will be required that the solution be obtained from regular initial conditions. In particular, the physical solution must have the initial data with regular center. For $0 < k < 1$, the central density parameter D_0, which is defined by

$$D_0 \equiv 4\pi\rho(t, 0)t^2, \tag{10}$$

well parametrizes the regular center of self-similar solutions, where t is chosen to be negative. For the regular center of self-similar solutions, the D_0 is finite and does not depend on t because

$$D_0 = \lim_{r \to 0+} 4\pi\rho(t, r)t^2 = \lim_{\xi \to -\infty} \frac{1}{2}\xi^2\eta(\xi). \tag{11}$$

For $0 < k < 1$, the ODEs have a singular point which is called a sonic point, at which the relative speed $|V|$ of the fluid element to the curve $\xi = \text{const}$ is equal to the sound speed $\sqrt{k}$. The matching condition beyond the sonic point results in nontrivial reduction of self-similar solutions. The reduction of the self-similar solutions was demonstrated by Ori and Piran (1990). If we require regularity, by which we mean the continuity of first derivatives, the possible solutions make a band structure in the space of all solutions. The D_0 axis is divided into allowed ranges and forbidden ranges (for which the solution terminates at the sonic point). If we further require analyticity (Taylor-series expandability), the possible solutions are discretized. The regular center is also analytic, where it should be noted that the analyticity at the center is defined with respect to the local Cartesian coordinates. However, only a discrete set of values for D_0 is allowed for the analyticity of the sonic point.

This complicated structure of physical solutions is due to the character of the sonic point as a critical point of the autonomous system, which was studied by Bicknell and Henriksen (1978). The sonic point regarded as a critical point of the autonomous system of ODEs after an appropriate transformation and is considered in the two-dimensional subspace of the whole state space. As a result, sonic points are classified into nondegenerate nodes, degenerate nodes, saddles and foci, among which the foci are unphysical. For every physical sonic point, there are two possible directions along which solutions can cross the sonic point, and there are two analytic solutions corresponding to these two directions. For a nondegenerate node, the two directions are called primary and secondary ones. For the former, there are one-parameter family of regular solutions including one of the analytic solutions, while for the latter, another analytic solution alone is possible. For a degenerate node, the two directions are degenerate and there are one-parameter family of regular solutions including the one analytic solution. For a saddle one, all that are regular at the sonic point are the two analytic solutions alone.

The numerical solutions with analytic initial data were investigated by Ori and Piran (1987, 1990) and Foglizzo and Henriksen (1993). This discrete set of solutions includes the flat Friedmann solution, general relativistic counterpart of the Larson-Penston solution, and that of the Hunter family of solutions. However, it is noted that the whole set of self-similar solutions with analytic initial data is still uncertain because of the numerical difficulty. Hereafter we refer to the general relativistic Larson-Penston solution and Hunter (a) solution as the Ori-Piran solution and Evans-Coleman solution for convenience. The most researched nontrivial self-similar solutions are the Ori-Piran and Evans-Coleman solutions. For the character of the sonic point, the Ori-Piran solution crosses a degenerate node along the secondary direction for $0 < k \lesssim 0.036$, a degenerate node for $k \simeq 0.036$ and a nondegenerate node along the primary direction for $0.036 \lesssim k < 1/3$ (Ori and Piran 1990, Harada 2001). Carr et al. (2001) investigated the detailed nature of the Evans-Coleman solution for $0 < k < 1$. The Evans-Coleman solution exists for $0 < k \leq 1$ (Brady et al. for $k = 1$). It crosses a saddle for $0 < k \lesssim 0.41$, a nondegenerate node along the secondary direction for $0.41 \lesssim k \lesssim 0.89$, a degenerate node for $k \simeq 0.89$ and a nondegenerate node along the primary direction for $0.89 \lesssim k < 1$. The Ori-Piran solution describes coherent collapse, while the Hunter family of solutions some oscillative collapse. The Hunter family of solutions are well labelled by the number of zeroes in their velocity fields. Ori and Piran (1987, 1990) also pointed out that a naked singularity forms in the Ori-Piran solution for $0 < k \lesssim 0.0105$, which may have some implication to cosmic censorship proposed by Penrose (1969, 1979). (See Harada, Iguchi and Nakao (2002) for recent works in naked singularity physics.) Actually, the general relativistic counterpart of the Hunter family of solutions with more than one oscillation can also have a naked singularity (Ori and Piran 1990, Foglizzo and Henriksen 1993). It is not clear whether nontrivial solutions other than the Evans-Coleman solution exist for every value of $k \in (0, 1]$, although numerical studies suggest that the structure of the solutions is common at least for $0 < k \leq 9/16$ (Ori and Piran 1990, Foglizzo and Henriksen 1993). The density profiles for these solutions are shown in Fig. 2.

For $k = 0$, i.e., a dust fluid, things are somewhat different. Ori and Piran (1990) studied self-similar dust solutions with regular center. There is one-parameter family of solutions. The regularity at the center requires that the central density parameter D_0 be $2/3$. The solutions are parametrized not by the central density parameter but by the central third-order derivative of the density field. However, this regular center is not analytic except for the flat Friedmann solution because the third-order derivative of the density field is not zero. Therefore, if we require that a physical solution has analytic initial data, the only allowed solution is the flat Friedmann solution. There is no sonic point for the dust case. See Carr (2000) for more general class of self-similar dust solutions.

4   Critical behavior

The critical behavior in gravitational collapse was numerically discovered by Choptuik (1993) in the spherical system of a massless scalar field. For a one-parameter family of initial data sets for this system, a critical value is found beyond which the evolution results in black hole formation (see Fig. 3(a)). For near-critical values, the critical behavior is observed. The solution first tends to a discrete self-similar solution, which is called a critical solution, and diverges away after that. For near-critical and supercritical
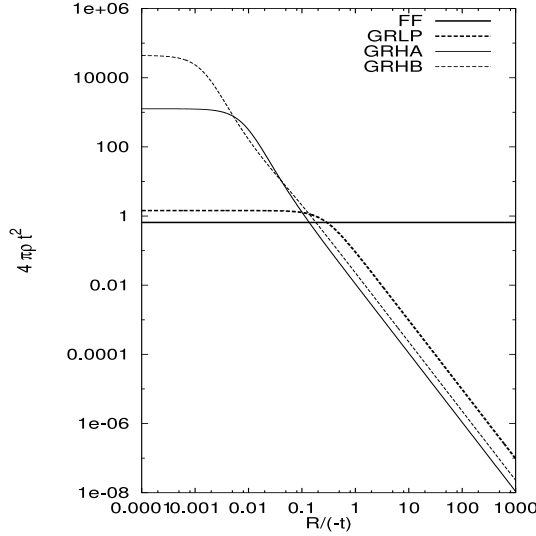
Figure 2: Density profiles for self-similar solutions with analyticity for $k = 0.01$. 'FF', 'GRLP' and 'GRHA' denote the flat Friedmann solution, general relativistic Larson-Penston solution or Ori-Piran solution and general relativistic Hunter (a) solution or Evans-Coleman solution, respectively.

collapse, the mass of the formed black hole follows a power law as

$$M_{BH} \propto |p - p^*|^{\gamma}, \tag{12}$$

where the constant γ is called a critical exponent (see Fig. 3(b)). The critical exponent for this system was estimated to be ≈ 0.37. After this discovery, a similar behavior was observed in various systems, such as gravitational waves (Abrahams and Evans 1993), a radiation fluid (Evans and Coleman 1994), a perfect fluid, a stiff matter (Neilsen and Choptuik 2000) and so on. Then, it has been found that the critical solution can be a discrete self-similar solution or continuous self-similar solution, depending on the system. For the fluid system including the radiation fluid system, the critical solution has continuous self-similarity. For the system of a radiation fluid, Evans and Coleman (1994) estimated the critical exponent to be ≈ 0.36 and also found that the critical solution obtained as a result of numerical simulations of the PDEs agrees very well with one of the continuous self-similar solutions which are obtained by integrating the ODEs. (That is why we refer to this self-similar solution as the Evans-Coleman solution in this article. We shall avoid the term 'critical solution' to refer to this solution because whether this solution is a critical solution may depend on the system, for example, whether we allow nonspherical perturbations or not, as we will mention later.) It has been subsequently found that the value of the critical exponent is not universal but depends on the parameter k which parametrizes the equation of state $P = k\rho$.

Koike, Hara and Adachi (1995) gave a clear picture for the critical behavior, which is called a renormalization group approach. According to this picture, the critical solution is assumed to be a self-similar solution with analytic initial data with a single unstable mode. We consider the time evolution of the solution with respect to the self-similarity coordinates $\tau \equiv -\ln|t|$ and $x \equiv -\ln|\xi|$. Then, the time evolution is regarded as a flow in the space of functions of x. The self-similar solution with a single unstable mode, which is a fixed point in the space of functions of x, has a stable manifold of codimension one. This implies that a one-parameter family of initial data generically has intersection with the stable manifold. The collapse which develops from the initial data set which is the intersection corresponds to the critical collapse, while near-critical collapse develops from the initial data set near the intersection. After a long time, the deviation of the near critical collapse from the critical solution is dominated by the single unstable mode, i.e.,

$$h(\tau, x) \approx H_{ss}(x) + (p - p^*)e^{\kappa\tau}F_{rel}(x), \tag{13}$$

where H_{ss} is the critical solution and κ and F_{rel} are the eigenvalue and eigenfunction of the single unstable
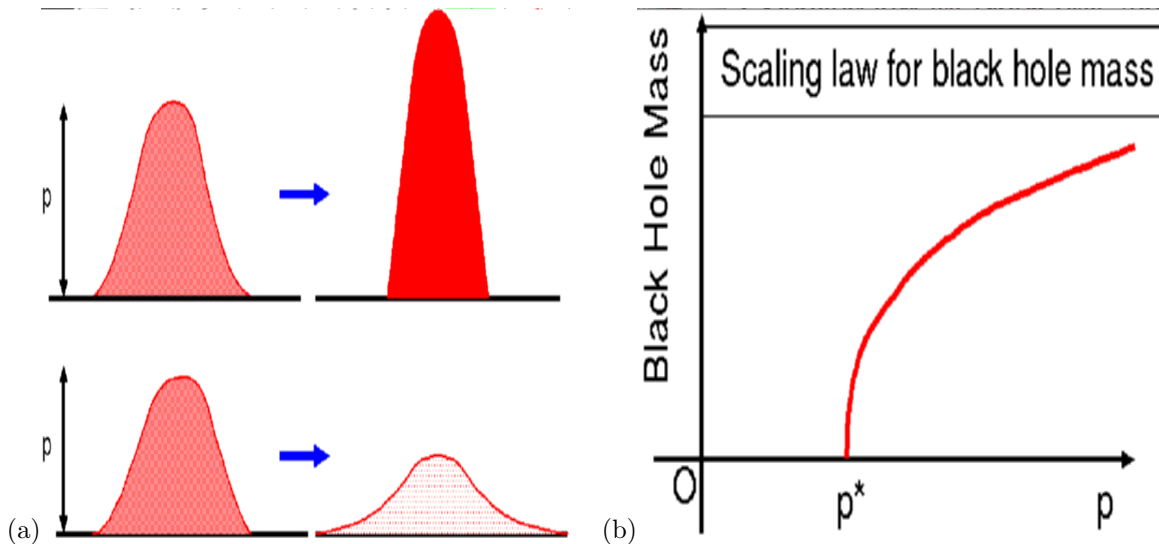
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Figure 3: Schematic pictures for the critical behavior in gravitational collapse. (a) indicates that the gravitational collapse has critical nature in general. In (b), the scaling law for the formed black hole is schematically displayed.

mode. Since the apparent horizon appears when the deviation becomes $O(1)$, the mass of the black hole is estimated as

$$M = O(r) = O(e^{-\tau}) \propto |p - p^*|^\gamma \tag{14}$$

where $\gamma = \kappa^{-1}$. The eigenvalue and eigenfunction of the unstable mode are determined by the linearized Einstein equation with the regularity condition both at the center and at the sonic point. Koike, Hara and Adachi calculated the critical exponent to be ≈ 0.3558019 by the eigenvalue analysis, which agrees very well with the estimated value ≈ 0.36 from the numerical simulation.

Critical phenomena in gravitational collapse have been observed and widely researched in a great variety of systems. The implication to the formation of primordial black holes was also discussed.

One of the interesting systems is the spherical system of a stiff matter. Neilsen and Choptuik (2000) observed the critical behavior in the stiff-matter system and found that the critical solution is a continuous self-similar solution. They also estimated the critical exponent to be 0.96 ± 0.02. Brady et al. (2002) confirmed that the Evans-Coleman solution for $k = 1$ has a single unstable mode and that the critical exponent derived from the eigenvalue analysis agrees well with that estimated from the result of numerical simulations of the PDEs. This critical behavior for the stiff-matter system seems to be totally different from that for the scalar-field system. On the other hand, a massless scalar field is equivalent with a stiff matter under certain circumstances (Madsen 1988, Brady et al. 2002). Brady et al. (2002) pointed out that the critical solution, which is discrete self-similar, for the system of a massless scalar field cannot be constructed from the stiff matter because the equivalence is broken.

See Gundlach (1998, 1999) for a recent review of critical phenomena in gravitational collapse.

5   Stability

The stability of self-similar solutions against spherically symmetric perturbations has called much attention in the context of the critical behavior in gravitational collapse since Koike, Hara and Adachi (1995) showed that the critical behavior is understood in terms of the behavior of the unstable mode of the critical solution.

There are two types of spherically symmetric perturbations, analytic modes and kink modes. First we consider the analytic modes. The perturbation for a dimensionless quantity h is written as

$$h(\tau, x) = h_0(x) + \delta h(\tau, x). \tag{15}$$

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After the linearization with respect to the perturbation, the perturbation δh is separated in two factors as

$$\delta h(\tau, x) = \epsilon e^{\kappa \tau} F(x). \tag{16}$$

For $0 < k \leq 1$, when the regularity condition is imposed both at the center and at the sonic point, the κ must take a discrete set of values. In other words, the allowed values for κ are determined as eigenvalues. If $\kappa > 0$, the mode is growing and referred to as an unstable mode. The stability of the Evans-Coleman solution has been investigated by Koike, Hara and Adachi (1995) for $k = 1/3$ and by Maison (1996), Koike, Hara and Adachi (1999), Harada and Maeda (2001) for more general value of $k \in (0, 1)$, and Brady et al. (2002) for $k = 1$. All these works have shown that the Evans-Coleman solution has a single analytic unstable mode for $0 < k \leq 1$. For the Evans-Coleman solution, which has a single unstable mode, the inverse of the eigenvalue of the unstable mode is interpreted as the value of the critical exponent, as we have seen in Sec. 4. The results are summarized in Fig. 4. The critical exponents directly estimated from the mass-scaling law obtained after the numerical simulations of the PDEs are also plotted.

The stability of other self-similar solutions with analytic initial data were investigated by Koike, Hara and Adachi (1999) and Harada and Maeda (2001). Then, it has been found that the Ori-Piran solution have no unstable mode, while the general relativistic counterpart of the Hunter-family solutions have unstable modes, the number of which agrees with the number of zeroes in the velocity field at least for the solutions examined there. Since the nonuniqueness of unstable modes implies that the codimension of the stable manifold is greater than one, it is expected that the approach to the self-similar solution is realized only as a result of fine-tuning more than one parameters simultaneously. This is the reason why the numerical simulations of the PDEs do not hit the self-similar solution with more than one unstable modes. On the other hand, since the Ori-Piran solution has no unstable mode, it is expected that generic evolution go towards this solution without any fine-tuning. In the next section, we will see that actually this is the case.

Next, for the kink modes, the existence in general relativity was expected by Ori and Piran (1990) and demonstrated by Harada (2001). For the analysis, we consider the initial perturbations which satisfy the following conditions: (For a while, we assume $t < 0$ and later will mention the case for $t > 0$.) (1) The initial perturbations vanish inside the sonic point. (2) The density field is continuous. (3) The density gradient field is discontinuous at the sonic point, although it has definite one-sided values inside and outside. See Fig. 5(a) for the initial setting of perturbations. The evolution of the density gradient at the sonic point turns out to be local, which simplifies the analysis. The analysis of the kink mode is possible in full order. Then, it is found that the stability against the kink mode is closely related to the classification of the sonic point, where the instability is defined by the blow up of the density gradient in a finite interval in terms of the time coordinate τ.

The result is as follows. All primary-direction nodal-point solutions, degenerate-nodal-point solutions, and saddle-point anti-transsonic solutions are unstable against the kink mode, while all secondary-direction nodal-point solutions, and saddle-point transsonic solutions are stable against the kink mode. The situation is reversed for $t > 0$ except for the case of degenerate-nodal-point solutions, which are unstable against the kink mode both for $t > 0$ and $t < 0$. Figure 5(b) shows the nonlinear evolution of the kink mode for a sonic point which is a nondegenerate node, as a typical example.

When we apply this criterion to the known solutions, we obtain the following results for the stability against the kink mode. The flat Friedmann (collapsing) solution is unstable for $0 < k < 1/3$ while stable for $1/3 \leq k < 1$. The Ori-Piran solution is stable for $0 < k \lesssim 0.036$ while unstable for $0.036 \lesssim k < 1/3$. The Evans-Coleman solution is stable for $0 < k \lesssim 0.89$ while unstable for $0.89 \lesssim k < 1$. Moreover, all regular but nonanalytic (collapsing) solutions are unstable.

The kink instability of the Evans-Coleman solution for $0.89 \lesssim k < 1$ will affect the critical behavior of a perfect fluid. However, Neilsen and Choptuik (2000) observed the critical behavior even for $0.89 \lesssim k \leq 1$ which is considered to be continuous extrapolation of that for $0 < k \lesssim 0.89$. It has not been explicitly shown but is suggested that the Evans-Coleman solution has a single analytic unstable mode even for $0.89 \lesssim k < 1$ from their result. Moreover, it was also reported that the Evans-Coleman solution has a single analytic unstable mode for $k = 1$ (Brady et al. 2002). About the reason why the kink instability did not affect the PDE calculations by Neilsen and Choptuik (2000), Harada (2001) speculated several possibilities.

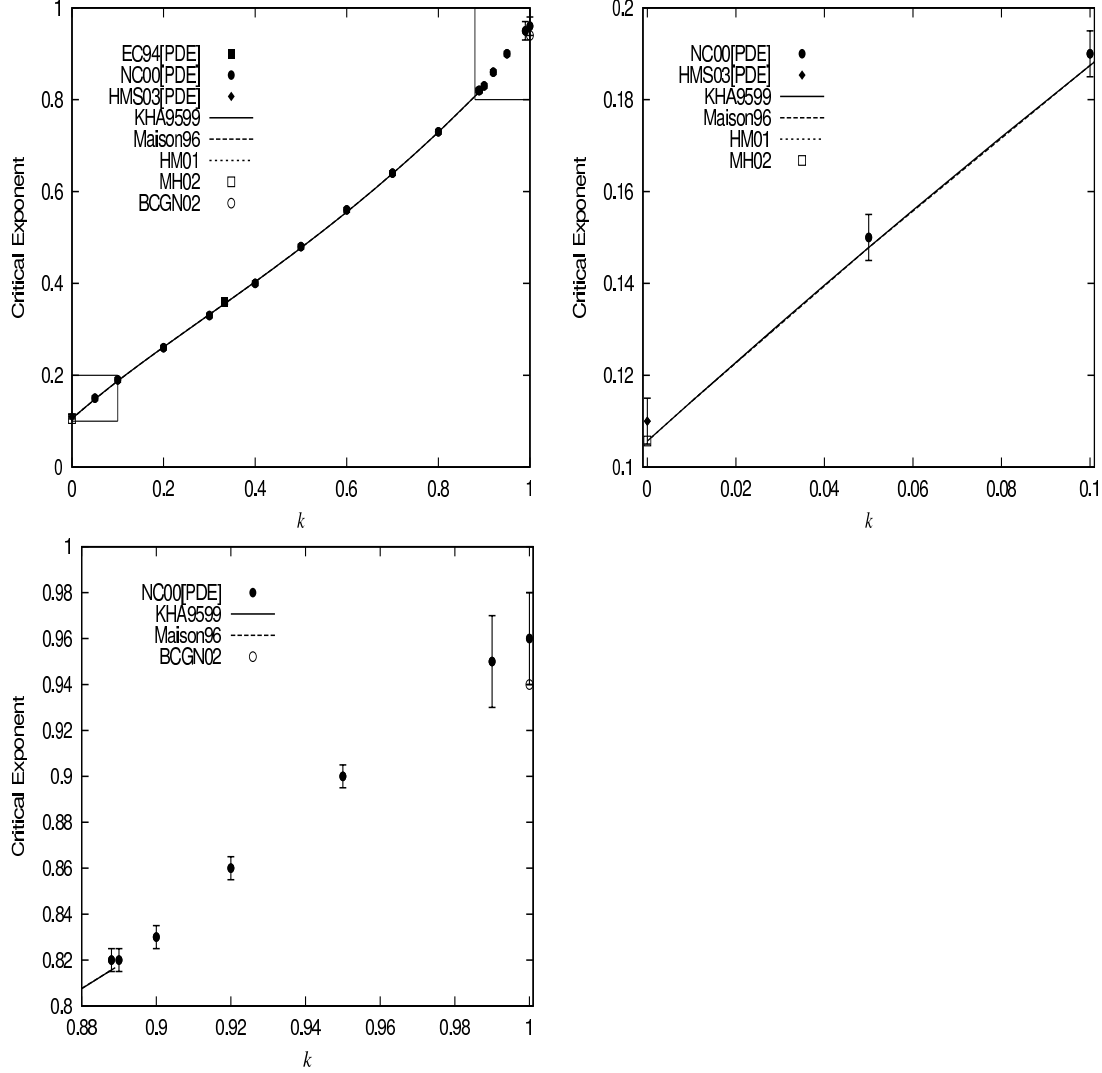The stability of the Evans-Coleman solution against nonspherical perturbations was investigated by

Figure 4: Critical exponents for various value of k. The plots with the label "[PDE]" were obtained as a result of numerical simulations of PDEs. All other plots were obtained by the eigenvalue analyses. The references are the following: EC94 = Evans and Coleman (1994), NC00 = Neilsen and Choptuik (2000), HMS03 = Harada, Maeda and Semelin (2003), KHA9599 = Koike, Hara and Adachi (1995, 1999), Maison96 = Maison (1996), HM01 = Harada and Maeda (2001), MH02 = Maeda and Harada (2002), BCGN02 = Brady et al. (2002). The error bars attached with the plots obtained from numerical simulations are determined from the significant digits when the numerical errors are not explicitly given in the reference. For KHA9599, Maison96 and HM01, the plots are replaced by curves because there are many plots and numerical errors are considered to be very small. Actually, these curves are hard to discriminate from each other. The plots for Newtonian case are plotted at $k = 0$ because the Newtonian system is regarded as the $k \to 0$ limit of general relativity.
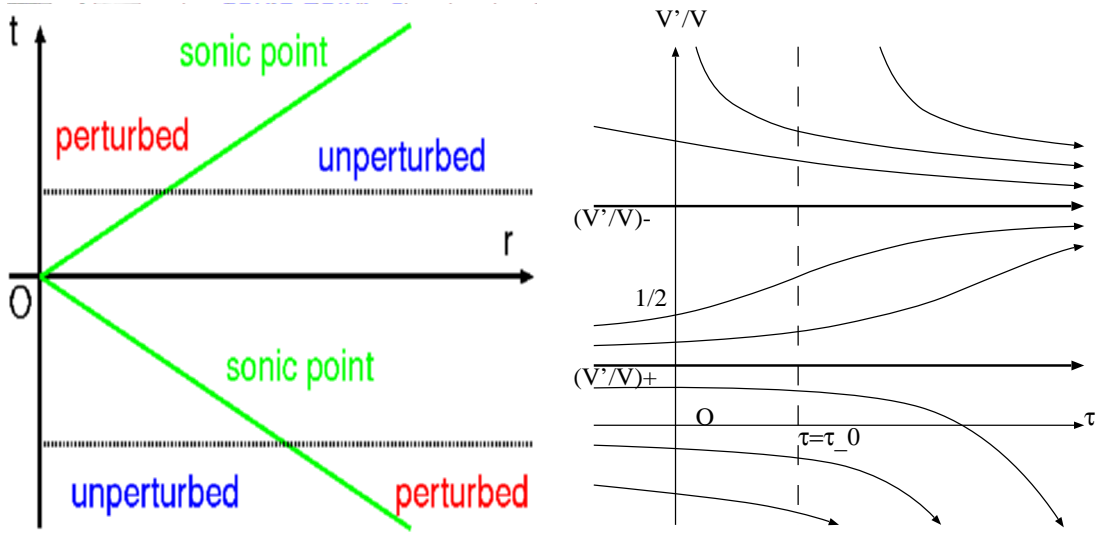
Figure 5: (a) Initial setting of kink-mode perturbations. For $t < 0$, the initial perturbation is put only outside the sonic point. (b) The nonlinear evolution of the perturbation is depicted. $+$ and $-$ denote the primary and secondary directions, respectively. The density gradient is proportional to the velocity gradient V'/V, where the prime denotes the partial derivative with respect to $x = -\ln|\xi|$. See Harada (2001) for details.

Gundlach (2002). The result is that the Evans-Coleman solution is stable against nonspherical perturbations for $1/9 < k \lesssim 0.49$ while is unstable for $0 < k < 1/9$ or $0.49 \lesssim k < 1$. For the stability of other self-similar solutions with analytic initial data against nonspherical perturbations, little has been known until now.

6 Attractor

As seen in the previous section, we have at least one self-similar solution which has no unstable mode. The solution is the Ori-Piran solution for $0 < k \lesssim 0.036$. From the discussion based on the renormalization group, which we have seen in Sec. 4, this solution seems to be an attractor which generic solution approach in the vicinity of the singularity formation. The numerical result obtained by Harada and Maeda (2001) strongly suggests that this is the case. In their numerical simulation of the PDEs, the collapse ended in singularity formation at the center for a certain subset of initial data sets, where the initial data sets were prepared without fine-tuning parameters. Thus obtained numerical solutions were compared with the self-similar solutions obtained by integrating the ODEs. Then, for the numerical solutions obtained by numerical simulations of the PDEs, the good continuous self-similarity is found and the profile of the density field agrees very well with the Ori-Piran solution, as seen in Fig. 6. It was confirmed that this convergence to the self-similar attractor does not depend on the choice of the initial density profile. The above is the results of numerical simulations for $0 < k \le 0.03$.

However, it is noted that we can consider initial data sets, from which the solution does not approach the Ori-Piran solution. For example, it is clear that the initial data set for the self-similar solutions other than the Ori-Piran solution does not show the approach to the Ori-Piran solution. Furthermore, if the central region is initially homogeneous and sufficiently compact, it can be shown that the solution approach the flat Friedmann solution around the center as the collapse goes towards the singularity formation. It is suggestive that the final fate of all these counterexamples is also described by self-similar solutions. Anyway, the result of numerical simulations strongly suggests that the set of the counterexamples to the convergence to the attractor is zero-measure in the space of initial data sets. These numerical results are well incorporated in the picture of renormalization flow; the critical solution as a saddle and the attractor solution as a sink in the renormalization flow of the function space.
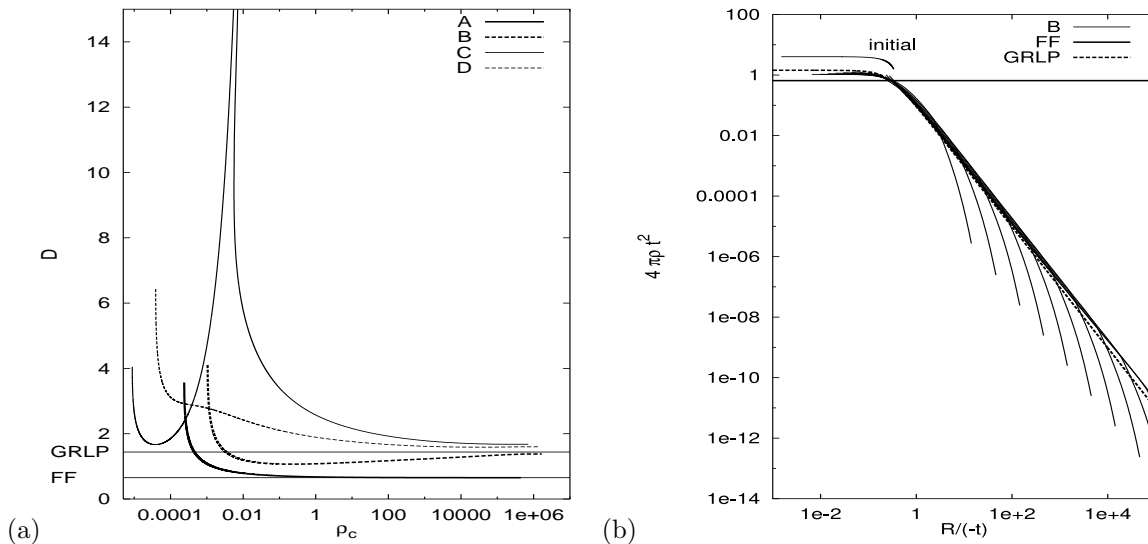
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Figure 6: In (a), the time evolution of the central density parameter D_0 is depicted as the collapse proceeds and the central density increases. In (b), the snapshots of density profiles is compared with the density profile of self-similar solutions. These are obtained by the numerical simulations of the PDEs for $k = 0.01$.

This is one realization of the self-similarity hypothesis in the spherically symmetric gravitational collapse. Moreover, since a naked singularity develops from the analytic initial data in the Ori-Piran solution for $0 < k \lesssim 0.0105$, this numerical study strongly suggests that the violation of cosmic censorship is generic in spherically symmetric spacetime. It should be noted that the numerical study based on the null formulation also obtained qualitatively the same result (Harada 1998).

From these considerations, it is expected that one of self-similar solutions may describe the asymptotic behavior of generic and/or special solutions in more general situations. Actually, several other examples are known to exhibit that a self-similar solution describes the asymptotic behaviors of more general solutions (Carr and Coley 1999). If the self-similarity hypothesis is true in a large variety of systems, not only will the analysis of the singularity and/or the asymptotic behavior of the spacetime be much simplified, but also should this be argued in the context of the structure of generic singularity (cf, Belinsky, Khalatnikov and Lifshitz 1970).

7   Newtonian case

Newtonian self-similar solutions have been investigated mainly in an effort to obtain realistic solutions of gravitational collapse leading to the star formation. Penston (1969) and Larson (1969) independently found a self-similar solution, which describes a gravitationally collapsing isothermal gas sphere. Here we call this solution the Larson-Penston solution. Thereafter, Hunter (1977) found a new series of self-similar solutions with analytic initial data and that the set of such solutions is infinite and discrete. Since he named the solutions (a), (b), (c), (d) and so on, here we call these solutions the Hunter (a), (b), (c) and (d) solutions and so on. Shu (1977) found other solution and discussed its astrophysical application. Whitworth and Summers (1985) pointed out the existence of a new family of self-similar solutions with loss of analyticity and found that these solutions make a band structure in the space of all solutions. When Ori and Piran (1990) extended these Newtonian self-similar solutions to general relativity, they showed that these Newtonian self-similar solutions are obtained as the limit of $k \to 0$ of general relativistic self-similar solutions. Even for a spherical gas system with the polytropic equation of state, it is possible to extend the definition of self-similar solutions and these solutions were studied by Goldreich and Weber (1980), Yahil (1983) and Suto and Silk (1988). However, here we concentrate on the case of isothermal gas.

There are two kinds of perturbation modes, the one is the analytic modes and the other is the kink modes, as we have seen in general relativistic case. For the former, the stability of self-similar solutions were studied by Hanawa and Nakayama (1997), Hanawa and Matsumoto (2000a, 2000b) and Maeda and Harada (2001) against spherically symmetric and nonspherical perturbations. For spherically symmetric perturbations, it is found that the homogeneous collapse solution and the Larson-Penston solution have no unstable mode, while the Hunter (a), (b), (c), (d) solutions have one, two, three and four unstable modes, respectively. It is expected all Hunter-family solutions have more than one unstable modes except for the Hunter (a) solution.

For the kink-mode perturbation, Ori and Piran (1988) showed that for $t < 0$ all primary-direction nodal-point solutions, degenerate-nodal-point solutions, and saddle-point anti-transsonic solutions are unstable against the kink mode, while all secondary-direction nodal-point solutions, and saddle-point transsonic solutions are stable against the kink mode. The situation is reversed for $t > 0$ except for the case of degenerate-nodal-point solutions, which are unstable against the kink mode both for $t > 0$ and $t < 0$. Actually, the above result is completely the same as in general relativity. Applying this criterion against the kink mode, it is found that the homogeneous (collapsing) solution is unstable while the Larson-Penston solution and the Hunter (a) solution are stable.

Since the above analysis indicates that the Hunter (a) solution has a single unstable mode, Maeda and Harada (2001) identified that the Hunter (a) solution is a critical solution in Newtonian collapse. The critical exponent calculated from the eigenvalue analysis is ≈ 0.10567, which is appropriate value as the $k \to 0$ limit of the critical exponent in general relativity. Moreover, the character of the Hunter (a) solution is very similar to that of the Evans-Coleman solution. For example, for both solutions, the central region contracts, the surrounding region expanding. From these observations, Harada and Maeda (2001) and Maeda and Harada (2001) identified that the Evans-Coleman solution is the general relativistic counterpart of the Hunter (a) solution.

Moreover, since the Larson-Penston solution has no unstable mode, it is regarded as an attractor solution. Actually, it has been considered and demonstrated that the Larson-Penston solution describes the generic collapse of an isothermal gas near the center since the discovery of this solution (Penston 1969, Larson 1969, Foster and Chevalier 1993, Tsuribe and Inutsuka 1999).

To find the critical behavior in Newtonian collapse and see the relation between the attractor and critical solutions, Harada, Maeda and Semelin (2003) performed the numerical simulations of the PDEs which govern the spherical system of an isothermal gas. The result is as follows: When a one-parameter of initial data sets is prepared, there is a critical value of the parameter. For the critical value the collapse tends to the Hunter (a) solution. For near-critical subcritical case, the collapse first approaches the Hunter (a) solution, thereafter diverges from that and finally disperses away. For near-critical supercritical case, the collapse first approaches the Hunter (a) solution, thereafter diverges from that and finally approaches the Larson-Penston solution. For the solution to approach the Larson-Penston solution, the parameter fine-tuning is not necessary but the parameter only has to be supercritical. The scaling law appears for the mass of the collapsed core for the near-critical supercritical case, while for the maximum density for the near-critical subcritical case. The critical exponents for these laws were estimated to be ≈ 0.11 and ≈ -0.22, which agree very well with the expected values, ≈ 0.10567 and ≈ -0.21134, respectively. Figure 7 shows the trajectory of solutions and the obtained scaling law for the collapsed core mass.

8   Summary

The general theory of relativity as well as Newtonian gravity admits self-similar solutions. This is due to the scale-invariance of gravity. The self-similar solutions are important not only because they are dynamical and inhomogeneous solutions easy to obtain but also because they may play important roles in the asymptotic behavior of more general solutions. We have seen that the stability analysis of self-similar solutions gives a unified picture of the critical behavior and convergence to attractor in gravitational collapse, which have been both observed in recent numerical simulations of the PDEs. Since both the critical behavior and convergence to attractor are not only in general relativity but also in Newtonian gravity, these two are considered to be common characters of gravitational collapse physics. In the above sense, gravitational collapse singles out self-similar solutions even if the initial condition given is complicated. Finally, this article is concluded by the schematic picture for gravitational collapse in Fig. 8.
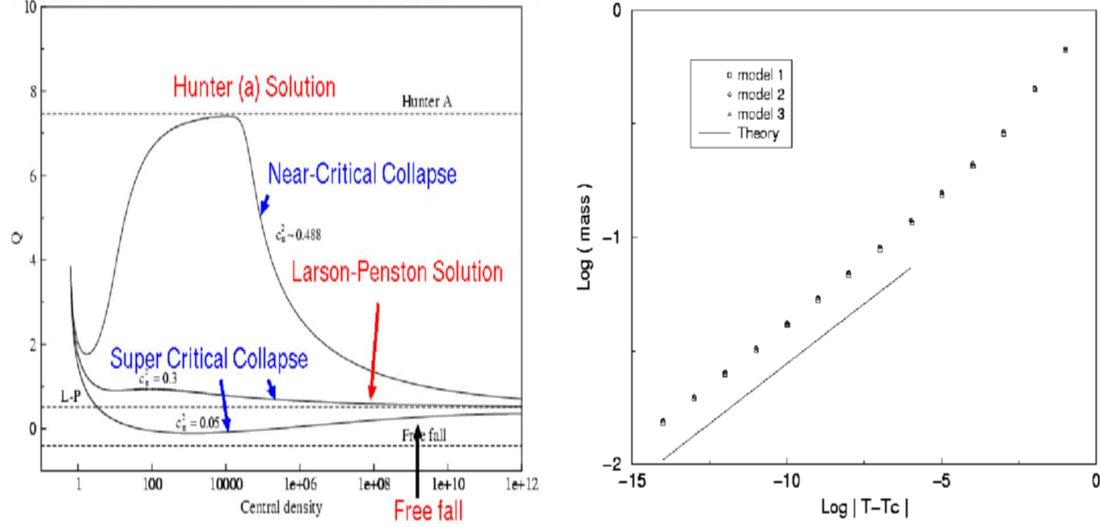
Figure 7: In (a), the time evolution of the central density parameter $Q = \ln D_0 = \ln[4\pi\rho(t,0)t^2]$ is depicted as a function of the central density. In (b), the scaling law for the collapsed core is depicted. In these simulations, the parameter of the one-parameter family of initial data sets is chosen to be the temperature of the system, the density profile fixed. For the details for definitions of quantities in this figure, see Harada, Maeda and Semelin (2003).
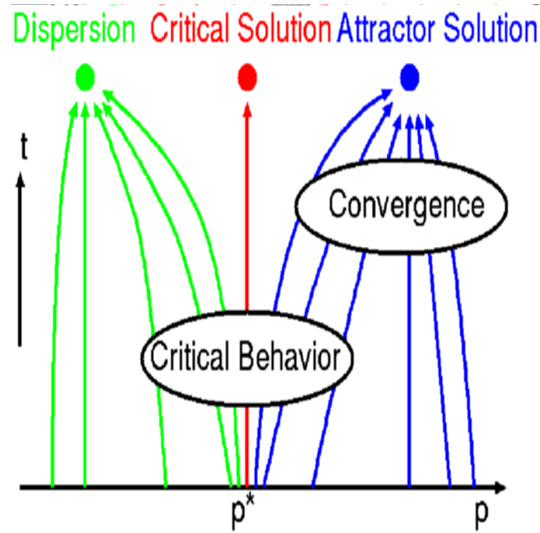

Figure 8: Schematic picture for the fate of gravitational collapse. This picture applies at least to the spherically symmetric perfect-fluid system with $0 < k \leq 0.03$ and to the Newtonian isothermal gas system.

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